Journal of Magnetic Resonance **155,** 278–292 (2002)
doi:10.1006/jmre.2002.2523, available online at http://www.idealibrary.com on IDEAL®

Non-CPMG Fast Spin Echo with Full Signal

Patrick Le Roux

General Electric Medical Systems, Marketing IRM, 283 rue de la Miniere, 78530 Buc Cedex, France

Received October 31, 2001; revised February 28, 2002

The standard Fast Spin Echo sequence used in MR imaging relies on the CPMG condition. A consequence of this condition is that only one component of the transverse magnetization can be measured. To counter this, some phase modulation schemes (XY, MLEV . .) for the pulse train have been proposed, but they are useful only over a very restricted range, close to π, of the refocusing pulse rotation angle. Some other solutions not relying on phase modulation have also been suggested, but they destroy one half the available signal. Revisiting the phase modulation approach, J. Murdoch ("Second SMR Scientific Meeting," p. 1145, 1994) suggested that a quadratic phase modulation could generate a train of classical echoes. We show here that indeed a quadratic phase modulation has a very suitable property: after an adequate change of frame, the dynamic of the system composed of all the protons situated in one pixel can be seen as stationary. If the parameter of the quadratic phase modulation is well chosen, it is then possible to put the dynamic system in a combination of two suitable states and obtain a signal identical to the signal of a classical spin echo, at least for nutation of the refocusing pulse higher than, approximately, two radians.

Key Words: **spin echo; Carr, Purcell, Meiboom, Gill; quadratic phase modulation.**

1. INTRODUCTION

1.1. Aim and Context of This Work

The Fast Spin Echo (FSE) sequence, or RARE (*1*), relies on the CPMG (Carr–Purcell–Meiboom–Gill) conditions (*2*) for generating a long train of spin echoes even when the rotation angle θ of the refocusing pulses is not equal to π. One of these conditions states that the initial magnetization after the excitation pulse must be aligned with the axis of the refocusing pulses. Any component of the magnetization perpendicular to this ideal direction is destroyed very rapidly after some echoes, as is depicted in Fig. 1. This figure demonstrates that the fast spin echo is very different from the classical Spin Echo experiment in which one component of the initial magnetization is maintained constant, while the perpendicular component alternates in sign with each echo.

This has some important practical consequences: procedures which rely on an auxiliary phase encoding, such as flow encoding and chemical shift separation, are not feasible with FSE. Other procedures such as diffusion weighting or T_2^* contrast, in which a parasitic phase modulation appears due to eddy currents, patient movement, or susceptibility effects, are also difficult to perform with FSE.

In the past decade, the MR imaging community has developed two related solutions to this problem (*3, 4*). The drawback of both solutions is that they give up, by principle, one half of the signal. Indeed as explained in Norris *et al.* (*3*), the sensitivity to the initial phase may be explained by the fact that the echo signal can be seen as the addition of two equal magnitude signals, with a phase which is equal to the phase of the initial magnetization for one signal, and opposite to that for the other signal. The Norris solution consists in eliminating, for instance by crusher gradients, one of these signals, resulting in an echo signal whose magnitude is indeed independent of the initial phase, but which is only one half of the signal obtainable when the CPMG condition is fulfilled. The Alsop solution (*4*) is more involved and reduces the signal oscillations which appears, in addition to the signal loss, in the Norris solution. But here, once again, one half of the pathways are discarded.

However, before that, Maudsley (*5*) and Guillon *et al.* (*6*) had a more ambitious target and tried to obtain a high, stable signal for both components of the initial magnetization. They used modulation schemes with pulses along the x and y axis (hence the generic name "XY" of these modulations) in a sequence determined by a recursive expansion procedure similar to the procedures developed for generating decoupling sequences. Unfortunately even the most sophisticated of these schemes cannot compensate for a large error of the refocusing pulse rotation angle. More recently, Murdoch (*7*) suggested that a quadratic phase modulation can act to preserve echo intensity in a spin-echo sequence.

We have explored this route (*8*) and applied it in practice (*9*). We want here to justify theoretically the use of a quadratic phase modulation. This was already presented (*10*), but the subject probably needs a more detailed explanation. Stated without too much mathematics, the theory relies on the equivalence of a quadratic phase modulation to a linear frequency sweep, a well-known principle in signal processing. Then, translating the coordinates at the same speed as this frequency sweep, one obtains a *stationary system*. This stationary system can be decomposed along its eigenvectors. If we can put the system state into the subspace spanned by the eigenvectors corresponding to certain

1090-7807/02 $35.00

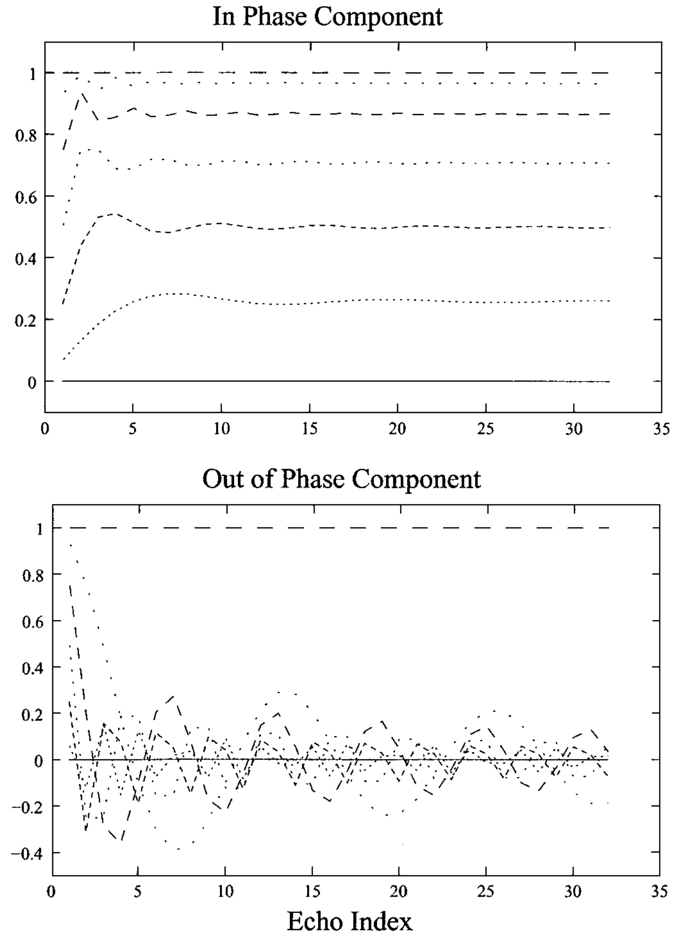

FIG. 1. Evolution of the transverse magnetization during a CPMG experiment, for echo number i ranging from 1 to 32 and rotation angle θ of the refocusing pulse varying from 0 to 180° with a 30° step. At the end of the excitation pulse, or echo number 0, the magnetization is assumed to have a magnitude of 1. At top, the initial magnetization is along the axis x of the refocusing pulse: the signal tends toward $\sin(\theta/2)$. At bottom, the initial magnetization is along y; for clarity, the magnetization has been multiplied in this case by $(-1)^i$.

eigenvalues (1, −1 in the classical O_3 representation, or $j, -j$, with $j = \sqrt{-1}$, in the spinor or SU_2 representation) we obtain some similarity with a classical spin-echo experiment. The signal is composed only of one constant part and one alternating part. These two signals have large and equal magnitude only for certain values of the frequency sweep rate. We give these values. We also determine a small phase sequence for the first few refocusing pulses which puts the system in the appropriate subspace. This gives the desired results down to a refocusing angle of 115°, i.e., far below what is possible with other known phase modulations. The reader primarily interested in the practical aspect of this work, rather than the mathematical basis, should read up to section "Setting the Receiver Phase" (Subsection 2.2), with Fig. 3 and then jump to the Subsection 3.6 "Resulting Optimal Phase Modulation."

1.2. Overview of the Work

Section 2 is devoted to the description of a train of echoes when the phase of the refocusing pulses varies according to an arbitrary law; after some preliminaries needed to define the action of crushing present in any fast spin echo sequence (Section 2.1) we answer the very practical question of how to set the receiving phase when the emission phase is varying (Section 2.2) and find that we can make the equations depend not on the emission phase itself but only on its first order difference (Section 2.3). In other words, we transform a phase modulation into a frequency modulation. But observing the system in a translating frame one can furthermore make the equations depend on the second order difference of the emission phase. Section 3 is devoted to the particular case of a quadratic phase modulation; because such a modulation has a second order difference which is constant, the equations describing the system dynamics become stationary (Section 3.1). Subsection 3.2 shows how to calculate the eigenfunctions of such a stationary system. In that subsection we do not choose a representation for the elementary rotations, but rather treat them as symbolic operators. To perform numeric computation we have the choice between the O_3 representation (our classical 3D space) or the SU_2 group where a rotation is expressed by two complex numbers. We prefer, here, this last and more compact solution and Subsection 3.3 describes the computation of the eigenfunctions of the stationary system in this representation. This task is performed in Subsection 3.3.2, after a succinct reminder of the SU_2 (or Quantum Mechanics) formalism in Subsection 3.3.1. In Subsection 3.3.3 we remined that the eigenfunctions being a complete orthogonal function basis, we can always express the system state in this basis. We are now in position, in Subsection 3.4, to tackle the problem on hand: obtaining a signal behavior similar to the perfect refocusing case even when the nutation angle is not π. We show that for obtaining this behavior, we must put the system in the subspace spanned by two eigenfunctions only. We show in Subsection 3.4.2 that this is the case for a perfect π pulse for which the initial flip pulse puts the system directly along the eigenfunction relative to the eigenvalue $-j$. Then, in case of a spurious phase introduced between the excitation pulse and the RF refocusing pulse train, we show that the system state stays in the subspace spanned by the two eigenfunction $j, -j$. In the next subsection, 3.4.3, we simply suppose that, in the absence of spurious phase, we have been able, after a preparation period lasting some echoes, to put the system state along the $-j$ eigenfunction, even when the nutation angle of the refocusing pulses is not π. Then, again, the system state will be in the subspace of eigenfunctions relative to $j, -j$, whatever the spurious dephasing introduced between the flip pulse and the train of echoes. But an eigenfunction is always defined up to an arbitrary phase only. We must choose this phase. In the next subsection, we take interest in the signal generated (this step would not have been necessary if we had used the O_3) and find out that the signal is composed of one constant part and one alternating part, the proportion between the two parts depending on the spurious phase introduced between the excitation pulse and the echo train. But if the constant part is given, the alternating part magnitude is dependent on the $-j$ eigenfunction phase factor. We maximize that alternating part, now entirely fixing the system state we will use as a target for the preparation period. But before embarking into that, we have still another parameter to determine: the second order finite difference of the quadratic phase modulation. That is the aim of Section 3.4.6, where we obtain this parameter by scanning a large set of possible values. It then suffices to determine the preparation period. This is presently done by a classical optimization programs, with a cost function defined in Section 3.5, the results of which are presented in Section 3.6 in the form of seven emission phase angles to apply to the first seven refocusing pulses. Section 4 gives an overview of the reconstruction process used in single shot acquisition. Section 5, Experimental Results and Discussion, summarizes the success but also the weakness of the present solution, and suggests some future developments.

2. REPRESENTATION OF A PHASE MODULATED SPIN-ECHO

2.1. Preliminary Remarks

Figure 2 shows the typical sequence that we will use. Only the excitation block and the first echo space (from the first refocusing pulse to the next) is shown; the subsequent echo spaces are

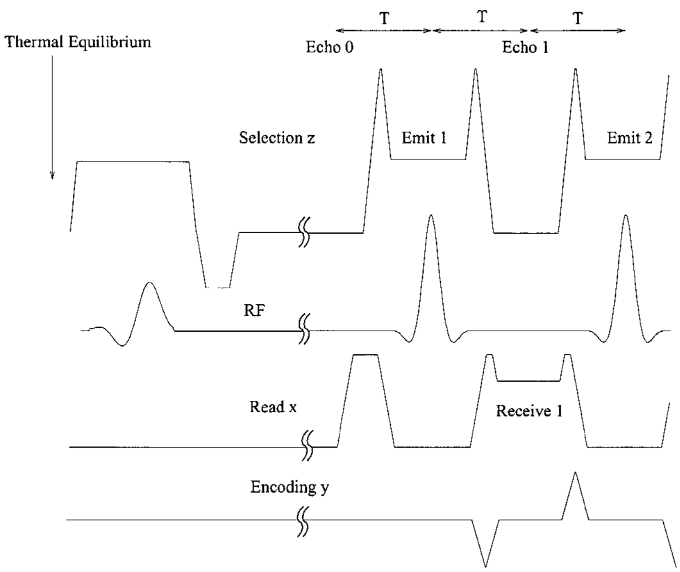

FIG. 2. The kind of sequences considered here. Nominally the center of the excitation pulse coincides with the reference time and the sequence is identical to the CPMG sequence, with gradients and B_0 integrals matching on each side of a refocusing pulse. Depicted here is the example of a T_2^* acquisition, where the time between the excitation pulse and the reference time has been elongated, resulting in an uncontrolled phase at the reference time. The rest of the sequence is not altered in any way, except that each refocusing pulse is emitted with a varying reference phase.

repeated identically to the first echo space, except, perhaps, for the encoding gradient lobes which vary. As can be seen, we do not make any change to the classical CPMG imaging sequence. Particularly, the integral of the gradients are kept equal on each side of the refocusing pulse, generating an echo at the middle of the echo space. As usual the exception to this symmetrical waveform is found on the phase-encoding direction where the phase-encoding lobe is compensated by an opposite lobe just before the next refocusing pulse. We denote by $2T$ the duration of one echo space. We call reference time, and also "echo 0," the position in time situated T before the center of the first refocusing pulse. In a standard CPMG experiment it is here that the center of the excitation pulse should be placed in order to eliminate the influence of the main field inhomogeneity ΔB_0. The depicted sequence is a T_2^* experiment and the period separating the center of the excitation pulse from the reference time has been elongated, giving at echo zero an unknown phase modulation of the object, denoted by χ. In other types of contrast, the phase modulation χ can be due to many other phenomena: for instance patient movement during a diffusion preparation inserted between the excitation pulse and the reference time. This phase modulation amounts to several 2π turns over the imaging volume, and thus the CPMG condition is violated completely resulting in a null signal at several positions in the volume.

We call K_x, K_z the integral of the gradients from the center of a refocusing pulse to the center of the subsequent read period, multiplied by γ the magnetogyric ratio (without any sign). We could add also a K_y integral, if a constant crusher lobe, in addition to the phase-encoding lobes, were to exist on the y axis. For a given geometric position, these gradient integrals induce, between the center of a refocusing pulse and the following echo, a precession ω that we can express in radians by

$$\omega(x, y, z) = K_x x + K_y y + K_z z + \gamma \Delta B_0(x, y, z)T. \quad [1]$$

Despite the fact that ω in this equation is an angle, we may take the liberty to call it a resonant angular frequency or even resonant frequency. This is to be understood as a resonant angular frequency which would give that angle of precession if applied continuously in the time unit T.

The gradient of the function $\omega(x, y, z)$ defines at each point a direction of crushing (which is, for large crusher gradients, virtually aligned with $\vec{K} = (K_x, K_y, K_z)^t$), and also a local crushing wavelength λ_c corresponding to a phase variation $\delta\omega$ of 2π. Apart from regions where the gradient of the main magnetic field becomes significant, this crushing wavelength is not very different from $\lambda_c = 2\pi/|K|$. The crusher gradients are adapted such that the crushing wavelength is small enough in comparison to the resolution of the image and the slice thickness. A ratio of one half between the crushing wavelength and the resolution is generally considered necessary. Also, as is usual when imaging a phased object, one must suppose that the phase modulation, $\chi(x, y, z)$ in our case, is not varying too fast in the length of one resolution, otherwise this phase modulation would induce, in the reconstructed image, a magnitude modulation as well. The crushing wavelength being smaller than the resolution, one then admits that at the time of echo zero, the magnetization is slowly varying in the scale of a crushing wavelength.

We intend to obtain, in spite of the phase χ, and although the refocusing angle θ is not π, a sequence of echoes in which one component of the initial transverse magnetization generates a constant signal, whereas the other perpendicular component gives an alternating signal. We note that this is possible because the echo signal is proportional to the *average*, over one crushing wavelength, of the transverse magnetization. One can thus imagine a distribution of the magnetization inside one crushing wavelength which on average mimics the behavior of a standard spin-echo with a π refocusing pulse, but of course with a reduced signal. We intend to realize this by phase modulating the train of refocusing pulses.

2.2. *Setting the Receiver Phases*

When varying the emission phases, i.e., the phase of the carrier during the RF pulses, the first question which arises in practice is how to set the phase of the carrier during the acquisition period. In Fig. 3 the phase of the first refocusing pulse is denoted ϕ_{x1}, the second one ϕ_{x2}, and so on. We suppose, for the moment, these emission phases to be given, and remark that these phases, with the addition of the initial phase χ of the magnetization, determine entirely the evolution of the magnetization in the subsequent echoes (for a constant rotation θ of the refocusing pulse). Hence the determination of the receiver phases which follows has the aim of rendering the expression for the received signal as simply as possible. We denote by ϕ_{r1} the carrier phase setting during the first read period (echo 1), ϕ_{r2} the phase during the second read period and so on. And, although there is no signal sampling at that time, we define a reference receiver phase ϕ_{r0} at echo 0. We would like (in the case of the nominal refocusing angle of π) the dynamics of the magnetization seen in the receiver frame to be simply the same as when no phase modulation is used, i.e., one component of the magnetization staying the same, the other one changing sign every other echo. This is realized, after arbitrarily setting the direction ϕ_{r0} of the magnetization elected to stay constant in the new frame, if we follow that magnetization under the influence of the successive π pulses. This readily fixes the set of receiver phases by

$$\frac{\phi_{r\,i} + \phi_{r\,i-1}}{2} = \phi_{x\,i}. \qquad [2]$$

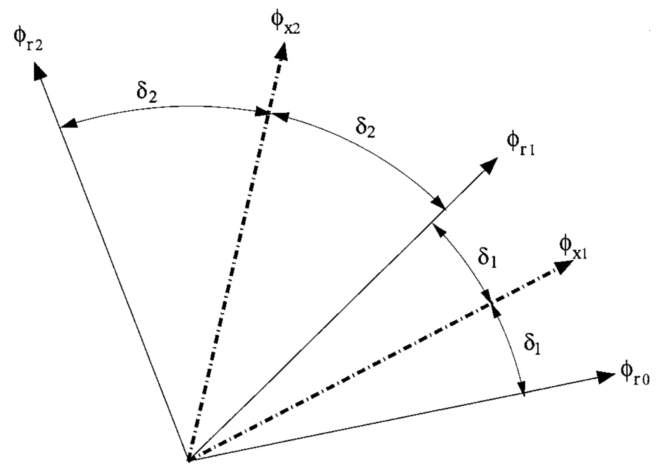

FIG. 3. If the phase of emission of each refocusing pulse is given, how should one set the phase of the receiver during each receive period? Such that, if the rotation angle of the refocusing pulse were π, an initial magnetization aligned with a chosen direction would generate a constant signal for every subsequent echoes.

Of course if the emission phases ϕ_x are given, the arbitrariness with which we can choose the initial receiving phase ϕ_{r0} presents a difficulty; one can add any sequence $\phi_{r0} \times (1, -1, 1, \ldots)$ to the receiving phase sequence. To solve this ambiguity one can try to eliminate or minimize that oscillation. In practice though, we will take as the driving quantity not the emission phases but the angle separating one acquisition phase from the next emission phase, which we note δ_i, and use this quantity to generate the two other phases (see Fig. 3). Hence

$$\phi_{x\,i} = \phi_{r\,i-1} + \delta_i, \qquad [3]$$

and

$$\phi_{r\,i} = \phi_{x\,i} + \delta_i. \qquad [4]$$

The only arbitrariness left is a constant phase ϕ_{r0} that we can add to both the ϕ_x and the ϕ_r. This is not relevant as it is just a change of the phase of the central carrier. The other advantage of using δ_i is that it can be merged with the rotation ω, as we see now.

2.3. *From Phase Modulation to Frequency Sweep*

In this subsection we write the rotation from one echo to the next in terms of symbolic rotation operators. $R_n(\phi)$ represents a rotation of angle ϕ around the unit vector (in the 3D space) $\vec{n}$, with some obvious notation like $R_z(\phi)$ representing a rotation around the axis z. We can restrict ourselves to taking a snapshot of the system at the echo time. From that knowledge we can deduce, by simple precession, the state at any moment of the echo space, up to just before or just after the RF pulses, which themselves are assumed to be applied instantaneously.

At a given position in space, at resonant angular frequency ω along the crushing direction, and between echo $i-1$ and emission i, the magnetization undergoes a rotation around z: $R_z(-\omega)$, seen in the central rotating frame. The minus sign comes from the fact that we are here considering protons, and we defined ω as a function of an unsigned magnetogyric ratio. But if we place ourselves in a frame with a reference axis (say x) aligned with the reception reference axis at echo $i-1$ and then with the emission reference axis at the center of pulse i, we see the magnetization rotating by $R_z(-\omega-\delta_i)$. Identically, and still viewing the magnetization in the new rotating frame whose axis x stays aligned with the emission and reception reference axis, the rotation undergone by the magnetization between emission i and echo i is: $R_z(-\omega-\delta_i)$. That leaves us with the characterization of the rotation induced by the RF emission when it is seen in a frame whose axis x is aligned with the B_1 field. Naively one is tempted to say that the rotation is $R_x(\theta)$. But one can argue that, the RF pulse being selective, this is only true at the center of the slice. Particularly, away from this center the axis of the rotation is tilted toward the axis z. However one can answer that this effect can always be taken into account by applying the same rotation around z on each side of the pulse. That precession can be included in δ_i, or even in ω if it is a constant, which is the case if the RF refocusing pulse itself is not varied. So the native vision is exact. Hence the rotation from echo $i-1$ to echo i in the new rotating frame is

$$\begin{aligned} R_{i-1,i} &= R_z(-\omega-\delta_i)R_x(\theta)R_z(-\omega-\delta_i), \\ &= \hat{R}(\omega+\delta_i,\theta). \end{aligned} \qquad [5]$$

In the last line of the equation we have written the echo to echo rotation in a way to emphasize the fact that the rotation is not a function of ω and δ_i separately but of $(\omega+\delta_i)$; i.e., it is a constant rotation

$$\hat{R}(\omega,\theta) = R_z(\omega)R_x(\theta)R_z(\omega), \qquad [6]$$

translated along the resonant frequency axis, or crushing direction by $-\delta_i$. We will call the rotation $\hat{R}(\omega,\theta)$, whose center is by convention at $\omega=0$ where the gradients have no effect, the central rotation. Hence the rotation $\hat{R}(\omega+\delta_i,\theta)$ is the central rotation translated to $\omega=-\delta_i$.

2.4. *One Step Further: The Sweep Velocity*

Let $\vec{v}_i$ be the state of the system, at resonant angle ω, at echo i (this state can be for instance the magnetization, i.e., three real values, at each ω if we use the classical representation of rotations). From Eq. [5], one can write:

$$\vec{v}_i(\omega) = \hat{R}(\omega+\delta_i,\theta)\vec{v}_{i-1}(\omega), \quad i=1\ldots. \qquad [7]$$

Let us define the quantity $\Omega_i=\omega+\delta_i$; this defines a new, translating, frame whose origin $\Omega_i=0$ coincides at each echo i with the center $\omega=-\delta_i$ of the rotation $\hat{R}(\omega+\delta_i,\theta)$. Then, let us apply a change of variable to try to express all quantities in terms of Ω_i.

For that let us first pose the following:

$$\vec{v}_0(\omega) = \vec{w}_0(\omega),$$

and

$$\vec{v}_i(\omega) = \vec{w}_i(\omega+\delta_i) = \vec{w}_i(\Omega_i), \quad i=1\ldots. \qquad [8]$$

To be able to use Eq. [8] for $i=0$, we will have to set $\delta_0=0$. Applying this change of variable to Eq. [7] one obtains

$$\vec{w}_i(\Omega_i) = \hat{R}(\Omega_i)\vec{w}_{i-1}(\Omega_i-\delta_i+\delta_{i-1}), \qquad [9]$$

where we see that only the first difference $\Delta_i=\delta_i-\delta_{i-1}$ of the δ intervenes. We consider now Ω_i to be the independent variable in place of ω, and we replace the notation Ω_i by Ω

$$\vec{w}_i(\Omega) = \hat{R}(\Omega,\theta)\cdot S(\Delta_i)\vec{w}_{i-1}(\Omega), \qquad [10]$$

where

$$\Delta_i = \delta_i-\delta_{i-1}, \quad i=2\ldots, \qquad [11]$$

and

$$\Delta_1 = \delta_1; \qquad [12]$$

after having defined $S(\Delta_i)$ as the translation operator by Δ_i. We will preferably use Eq. [10] rather than Eq. [7]. Indeed, being given $(\phi_{r0}, \Delta_1, \Delta_2, \Delta_3\ldots)$ is equivalent to being given $(\phi_{r0}, \delta_1, \delta_2\ldots)$. Also, the signal at an echo time is the average value of the transverse magnetization over one crushing wavelength, and thus it is the same whether it is calculated from $\vec{v}_i$ or $\vec{w}_i$, these two values being only a translation from each other (and periodic with a period of one crushing wavelength). So we can directly use $\vec{w}_i$ to calculate the signal at echo i and can forget $\vec{v}_i$ altogether.

3. QUADRATIC PHASE MODULATION

3.1. *A Stationary System*

Let us try a constant sweep velocity $\Delta_i=\Delta$. Using Eqs. [11], [3], and [4] gives $\delta_i=\Delta i$, $\phi_{r\,i}=\Delta i(i+1)$, $\phi_{x\,i}=\Delta i^2$. Let us change Δ_1 and take for instance $\Delta_1=\Delta+\delta$; this gives $\delta_i=\Delta i+\delta$, $\phi_{r\,i}=\Delta i(i+1)+2\delta i$, $\phi_{x\,i}=\Delta i^2+(2i-1)\delta$. Finally, we can change ϕ_{r0} to add an identical constant to all phases. Hence we can simulate the action of any quadratic modulation of the emitter phases, with our three parameters Δ, Δ_1, ϕ_{r0}.

In all cases, at least after the first echo ($i>1$), when we place ourselves in the frame translating at a constant rate $-\Delta$ between each echo, $\Omega=\omega+\Delta i$, we obtain a system in Eq. [10] which is stationary,

$$\vec{w}_i(\Omega) = [\hat{R}(\Omega,\theta)\cdot S(\Delta)]\,\vec{w}_{i-1}(\Omega). \qquad [13]$$

3.2. *Constructive Definition of Eigenvectors*

Having in hand a stationary system, it is natural to try to represent it by its eigenvectors. Let us try to build one eigenvector $\vec{u}$ corresponding to a given eigenvalue λ (supposing we know the set of eigenvalues). One must realize that Eq. [13] corresponds to a dynamic system with very large dimensions, and we will rather use the term eigenfunction rather than eigenvector. Let us choose one particular frequency Ω_0 and suppose that the eigenfunction at that point is $\vec{u}(\Omega_0)$. This is a three-dimensional real vector if one works with magnetization (O_3 formalism), but will be a two-dimensional complex vector if we use spinors (SU_2 formalism). In this subsection we do not yet make the choice between the two representations. How will $\vec{u}(\Omega_0)$ be transformed by Eq. [13] at the next echo? First, it will be translated by Δ, hence it will be now seen at position $\Omega_1 = \Omega_0 + \Delta$; then it will undergo the rotation $\hat{R}$ at that position, becoming $\hat{R}(\Omega_1)\vec{u}(\Omega_0)$. But, by definition of an eigenfunction relative to an eigenvalue λ, the new value of the state vector at that position Ω_1 should be equal to $\lambda\vec{u}(\Omega_1)$. Hence we must have

$$\lambda\vec{u}(\Omega_1) = \hat{R}(\Omega_1)\vec{u}(\Omega_0). \qquad [14]$$

After dividing this equation by λ, one is able to determine the value of the eigenfunction $\vec{u}(\Omega_1)$, at $\Omega_1 = \Omega_0 + \Delta$, if its value at Ω_0 is known. But continuing the same procedure one can find its value at $\Omega_2 = \Omega_1 + \Delta$, and so on. Thus all the values of the eigenfunction situated on the "comb" of frequencies

$$\Omega_l = \Omega_0 + l\Delta \qquad [15]$$

can be deduced from $\vec{u}(\Omega_0)$, by

$$\vec{u}(\Omega_l) = \frac{1}{\lambda^l}\mathcal{R}_l\vec{u}(\Omega_0); \qquad [16]$$

with the definitions

$$\mathcal{R}_l = \mathcal{R}(\Omega_l) = \hat{R}(\Omega_l)\hat{R}(\Omega_{l-1})\ldots\hat{R}(\Omega_1); \qquad [17]$$

or, in recursive form,

$$\mathcal{R}(\Omega_l) = \hat{R}(\Omega_l)\mathcal{R}(\Omega_{l-1}). \qquad [18]$$

Let us now suppose that Δ is restricted to be a rational number,

$$\Delta = 2\pi\frac{n}{d}; \qquad [19]$$

with n and d two integers without common divisor. Hence the successive frequencies of the comb defined in Eq. [15] become

$$\Omega_l = \Omega_0 + 2\pi\frac{n}{d}l, \qquad [20]$$

with the remark that the integer nl in the above equation is to be taken modulo d. The integers generated by $k = nl\ (d)$ span in a one to one manner the set of integer $1\ldots d$. Notice also that the sequence is cyclic, with period d. So after having applied our previous procedure d times, $k = nl$ has spanned the whole set of d values $k = 1\ldots d$, and for $l = d$ the corresponding frequency Ω_d is again equal to Ω_0. Hence $\vec{u}_0$ is not arbitrary and must satisfy $\vec{u}(\Omega_d) = \vec{u}(\Omega_0)$:

$$\vec{u}(\Omega_0) = \frac{1}{\lambda^d}\mathcal{R}(\Omega_d)\vec{u}(\Omega_0). \qquad [21]$$

We will call $\mathcal{R}(\Omega_d)$ the "cycle rotation." It is indeed the rotation undergone by the particular magnetization situated at the resonant frequency $\omega = \Omega_0$ during one whole cycle of d successive rotations. $\mathcal{R}_d$ is a 3×3 real orthogonal matrix in the O_3 formalism or a unitary 2×2 complex matrix in the SU_2 formalism. From Eq. [21] one deduces that $\vec{u}(\Omega_0)$ must be one of the eigenvectors of $\mathcal{R}_d$. Let μ be one of the eigenvalues of this matrix ($|\mu| = 1$), and $\vec{u}_\mu$ its associated eigenvector. Equation [21] implies $\lambda^d = \mu$. From μ we thus can generate d different possible eigenvalues of the system in Eq. [13],

$$\lambda_m = (\mu)^{1/d}e^{j\frac{2\pi}{d}m}, \quad m = 1\ldots d. \qquad [22]$$

The corresponding eigenfunctions are calculated according to Eq. [16],

$$\vec{u}_{\mu,m}(\Omega_l) = \lambda_m^{-l}\mathcal{R}_l\vec{u}_\mu. \qquad [23]$$

Hence the eigenfunctions of the system in Eq. [13] are easily constructed numerically, on the comb $\Omega_k = \Omega_0 + k\frac{2\pi}{d}$. One notes that, as usual, each eigenfunction is still undetermined by a global phase factor applied for instance to $\vec{u}_\mu$, but all the components $\vec{u}(\Omega_0 + 2\pi k/d)$, $k = 1\ldots d$ are now linked to each other. However the eigenfunctions on two different combs, each comb corresponding to a distinct reference frequencies $\Omega_0 \in [0, 2\pi/d]$, are independent of each other and could, in principle, be multiplied by two unrelated phase factors. We actually will set these phases such that the resulting function of Ω be the most continuous possible.

3.3. *Computation of the Eigenvectors in SU_2*

From this point on we will use the SU_2 formalism, which more efficiently carries the same information as the classical representation using magnetization (the O_3 approach).

3.3.1. Quick overview of the spinor formalism. The reader not proficient with the SU_2 formalism should consult a textbook on quantum mechanics (*11*), application of spinor or Cayley Klein (CK) parameters (*12, 13*), or even quaternions (*14*). We give here only the necessary notations. We use SU_2 matrices, denoted Q, and density (or vector) matrix denoted σ, of the form, respectively

$$Q(\alpha,\beta)=\begin{bmatrix}\alpha & -\bar{\beta}\\ \beta & \bar{\alpha}\end{bmatrix},\quad \sigma(z,X)=\begin{bmatrix} z & \bar{X}\\ X & -z\end{bmatrix}. \qquad [24]$$

The density matrix $\sigma(z, X)$ represents a vector in the classical 3D space with projection z along the z axis, and $X = x + jy$ projection in the x, y plane. Under a rotation this 3D vector is transformed into another vector z', X'. This rotation is expressed with a multiplication on the left by an SU_2 matrix and a multiplication on the right by the conjugate transpose of this SU_2 matrix

$$\sigma(z', X') = Q\sigma(z, X)Q^*. \qquad [25]$$

The most general rotation of angle Θ around an axis $\vec{n}$ whose coordinates along z and the xy plane are, respectively, n_z and $n_{xy} = n_x + jn_y$, is represented by $Q_n(\Theta) = Q(\mathcal{C} - j\mathcal{S}n_z, -j\mathcal{S}n_{xy})$ with $\mathcal{C} = \cos(\Theta/2)$ and $\mathcal{S} = \sin(\Theta/2)$. In this very general case, the explicit writing of the full SU_2 matrix, rather than the use of a notation Q may be more appropriate because, by separating the real part of the parameter α, one can highlight the vector matrix representing the axis of the rotation

$$Q_n(\Theta) = \mathcal{C}1 - j\mathcal{S}\sigma(n_z, n_{xy}). \qquad [26]$$

3.3.2. Spinor computation of the eigenvectors. We now represent all rotations in the SU_2 form and use the same notation for the symbolic operator and its SU_2 matrix expression (with, when it is necessary to define the parameters values, $Q(\alpha, \beta)$ being understood as a short cut for the full SU_2 matrix as defined by Eq. [24]). In SU_2 the central rotation defined in Eq. [6] and used in Eqs. [10], [13] is written

$$\hat{R}(\Omega,\theta)=\begin{bmatrix} ce^{j\Omega} & -js\\ -js & ce^{-j\Omega}\end{bmatrix}, \qquad [27]$$

with $c = \cos(\theta/2)$, $s = \sin(\theta/2)$. Thus, each elementary rotation $\hat{R}(\Omega_l)$ in Eq. [17] is represented by $\hat{R}(\Omega_l) = Q(c\exp(j\Omega_l), -js)$.

Supposing that $\mathcal{R}_l$ is represented by a matrix $\mathcal{R}(\Omega_l) = Q(\alpha_l, \beta_l)$, its computation is performed recursively by the equivalent of Eq. [18] in spinor form. This way, we obtain the cycle rotation $\mathcal{R}(\Omega_d)$ in Eq. [21]. The diagonalization of $\mathcal{R}(\Omega_d)$ is easy: one begins by writing it in the form $\mathcal{R}(\Omega_d) = \mathcal{C}1 - j\mathcal{S}\sigma_V$; then the vector matrix σ_V is factored into $\sigma_V = U\sigma_z U^*$, with $\sigma_z = \sigma(1, 0)$. One verifies immediately that the first column of U, which we denote $\vec{u} = [u, v]^t$, is the eigenvector of $\mathcal{R}_d$ with an eigenvalue $\mu = \mathcal{C} - j\mathcal{S}$; the second column of U (noted $\vec{u}_\perp$) is the eigenvector with an eigenvalue $\bar{\mu} = \mathcal{C} + j\mathcal{S}$. The detailed computation in terms of the CK parameters α_d, β_d of $\mathcal{R}_d$ is given in Appendix A. Further, we show in Appendix B, by using a polynomial representation (*15*) that, if d is odd, and for sufficiently high cycle energy

$$d \times \theta^2 \gg 1, \qquad [28]$$

with θ, the rotation angle of the refocusing pulse, expressed in 2π turns, we have $C \simeq 0$. That means that the rotation angle of the cycle is close to π, and hence the eigenvalues are $\mu \simeq -j$ and $\mu \simeq j$. Note that one can demonstrate in a similar, but somewhat more involved, manner that if d is a large even number, the cycle rotation angle is 2π (transparent rotation). This case is not considered in this article any further, and from now on, d is supposed odd (the results for two consecutive d, hence odd and even, can be made similar; but the case d "even" necessitates dealing with degenerated eigenstates). From all the values of λ_m, or $\bar{\lambda}_m$ we chose the one which is the closest to $-j$. This implies choosing the eigenvector of the cycle rotation corresponding to $\mu = -j$ if $d = 4q+1$, or to $\mu = j$ if $d = 4q-1$. We still call $\vec{u}$ the corresponding cycle eigenvector, after eventually having made the necessary swap between the two column vectors $\vec{u}$, $\vec{u}_\perp$ of the matrix U. And then we choose $m = 0$ in Eq. [22], and use Eq. [23] with $\lambda_m = -j$ to obtain the "main" eigenfunction that we denote $\vec{u}_{-j}(\Omega_l)$. We could similarly generate the eigenfunction $\vec{u}_j(\Omega)_l$, but it is easily found that the two vectors are deducible from each other as the two column vectors of an SU_2 matrix. For the rest of this article we will denote by $U_0(\Omega_l)$ the matrix composed of the concatenation of the two eigenfunctions (2D vectors) $\vec{u}_{-j}$, and $\vec{u}_j$, i.e., $U_0(\Omega_l) = [\vec{u}_{-j} \mid \vec{u}_j]$, or in terms of the individual scalar components of $\vec{u}_{-j} = [u_l, v_l]^t$,

$$U_0(\Omega_l) = [\vec{u}_{-j} \mid \vec{u}_j] = \begin{bmatrix} u_l & -\bar{v}_l\\ v_l & \bar{u}_l\end{bmatrix}. \qquad [29]$$

One could generate the other $2d-2$ eigenfunctions for $m \neq 0$ by multiplying $\vec{u}_{-j}(\Omega_l)$, and $\vec{u}_j(\Omega_l)$ with the scalar $\exp(-j\frac{2\pi}{d}ml)$ but, to keep a SU_2 matrix structure, it is logical to group together the two eigenfunctions corresponding to two eigenvalues which are complex conjugated to each other. All eigenfunctions can then be obtained, from U_0, by simple matrix multiplication,

$$U_m(\Omega_l) = U_0(\Omega_l)R_z\left(\frac{2\pi}{d}ml\right), \qquad [30]$$

for the values of $m = 1 \ldots d-1$.

Also each eigenfunction is determined with an arbitrary phase factor. By convention, and according to Appendix A, we choose the first component (u_0), of $\vec{u}_{-j}$, at the reference frequency of the comb Ω_0, to be real. We can then multiply the whole eigenfunction $\vec{u}_{-j}(\Omega_l)$ by a phase factor $\exp(-j\psi_0/2)$. By convention we then multiply $\vec{u}_j(\Omega_l)$ by $\exp(j\psi_0/2)$, such that the new $U_0(\Omega_l)$ is still a SU_2 rotation matrix.

3.3.3. Decomposition of the system state on the eigenfunctions. We are now in possession of $2d$ eigenfunctions by which we can express any distribution of $2D$ complex vector (spinor) distribution $\vec{w}(\Omega_l)$ on the d frequency point Ω_l; but again to maintain the SU_2 type of symmetry, we will prefer to say that we can decompose any SU_2 matrix distribution $W(\Omega_l) = [\vec{w}(\Omega_l) \mid \vec{w}_\perp(\Omega_l)]$ by

$$W(\Omega_l) = \sum_{m=0}^{d-1} U_m(\Omega_l)\Gamma_m. \qquad [31]$$

Each "coefficient" matrix Γ_m is expressibly by two complex numbers $\Gamma_m = Q(\gamma_m, \sigma_m)$. In the following subsection we will consider $W(\Omega)$ to represent the rotation applied, at frequency Ω, by all the actions, RF pulse, gradient lobes, eventually spurious de-phasing χ, starting with the thermal equilibrium. That is to say we include the $\pi/2$ excitation pulse in the rotation. We call this rotation "global" rotation. Hence, supposing that we know, on the comb of frequencies as defined in Eq. [20], the global rotation $W(\Omega_l)$, we can decompose it in the form of a sum such as Eq. [31]. Due to the orthogonality of the eigenfunctions, these coefficient matrices (or the scalar coefficients γ_l, σ_l) are easily found,

$$\Gamma_m = \frac{1}{d}\sum_{l=0}^{d-1} R_z\left(-\frac{2\pi}{d}ml\right)U_0^*(\Omega_l)W(\Omega_l). \qquad [32]$$

3.4. Mimicking a Perfect Refocusing Pulse

3.4.1. Restriction to a subspace. We will not need the general decomposition in this paper. We indeed suppose that at a certain echo numbered p (p standing for "preparation"), the SU_2 matrix representing the global rotation is expressible uniquely by U_0, and the above decomposition in Eq. [31] is reduced to one term only,

$$W(\Omega_l, p) = U_0(\Omega_l)\Gamma. \qquad [33]$$

Note that the RF pulse train from the RF refocusing pulse 1 to the RF pulse p is to be determined, but starting from the echo p we suppose that the modulation of the RF train becomes pure quadratic with the sweep velocity Δ. Hence the system follows Eq. [13], and as U_0 is composed of the eigenfunctions relative to the eigenvalues $j, -j$ the subsequent evolution of the global rotation at echo $p + i$ is

$$W(\Omega_l, p + i) = U_0(\Omega_l)R_z(i\pi)\Gamma. \qquad [34]$$

To find out what the meaning and value of the coefficient matrix Γ is, let us have a look first to the special case of a perfect refocusing pulse equal to π.

3.4.2. Restriction to a subspace, the perfect refocusing pulse case. When the refocusing pulse is a π pulse along x, the central rotation $\hat{R}(\Omega)$ in Eqs. [10], [13], and [27] is represented by the SU_2 matrix $Q(0, -j)$ and has no Ω dependence. The eigenvalues of such a matrix are indeed $-j$, j, with the matrix of eigenvectors being $U_0 = Q(1, 1)/\sqrt{(2)} = R_y(\pi/2)$. This means that right after the excitation pulse, which is represented by $R_y(\pi/2)$, without any further preparation, the rotation from thermal equilibrium verifies Eq. [33] with $\Gamma = 1$. But how is this result changed if a rotation by χ around z is inserted between the excitation pulse and before echo zero? Then the rotation from thermal equilibrium to echo 0 is $W_0 = R_z(\chi)R_y(\pi/2)$. But using the fact that $R_y(-\pi/2)R_z(\chi)R_y(\pi/2) = R_x(-\chi)$ one finds that W_0 can again be expressed in the form of a combination of the eigenvectors U_0: $W_0 = R_y(\pi/2)R_x(-\chi) = U_0Q(\cos(\chi/2), j\sin(\chi/2))$. Hence the global rotation at echo zero is expressible as in Eq. [33] with $\Gamma = R_x(-\chi)$.

3.4.3. Restriction to a subspace, generalization to other nutation angles. For generalization we can suppose that, by careful design, we made the first p refocusing pulses, combined with the initial excitation pulse $R_y(\pi/2)$, induce a global rotation at echo p which is exactly equal to $U_0(\Omega_l)$. Hence, in the absence of spurious dephasing χ, the rotation $P(\Omega_l)$ induced by the first p refocusing pulses and dephasing periods, satisfies,

$$U_0(\Omega_l) = P(\Omega_l)R_y(\pi/2). \qquad [35]$$

Supposing now there is a precession χ inserted right after the excitation pulse, the global rotation at echo p is $W_p(\Omega_l, p) = P(\Omega_l)R_z(\chi)R_y(\pi/2)$. Eliminating P between these last two relations, we find again, after the same manipulation of $R_z(\chi)$, that $W(\Omega_l, p) = U_0(\Omega_l)R_x(-\chi)$. And the subsequent evolution at echo $p + i$ is given by Eq. [34]: $W(\Omega_l, p + i) = U_0(\Omega_l)R_z(i\pi)R_x(-\chi)$.

3.4.4. Restriction to a subspace, the measured signal. We now use Eq. [25] with $\sigma_z = \sigma(1, 0)$, and we find that the magnetization at echo $p + i$ is

$$\sigma(\Omega_l, p + i) = U_0(\Omega_l)\sigma(\cos(\chi), (-1)^i j\sin(\chi))U_0^*(\Omega_l).$$

We are interested principally by the transverse component of this vector. Remembering the definition of the scalar CK parameters u_l, v_l of U_0 shown in Eq. [29], the transverse magnetization component can be expressed as

$$X(\Omega_l, p+i) = \cos(\chi)(2\overline{u_l}v_l) + (-1)^i j\sin(\chi)\left(\bar{u}_l^2 + v_l^2\right). \qquad [36]$$

The signal we collect is the integral in one crushing pixel, so the signal $\hat{X}(\Omega_0)$ coming from the considered comb $\Omega_l = \Omega_0 + l\Delta$ is obtained by summation over the index l. Performing this summation in Eq. [36] suggests the use of the following quantities,

$$I(\Omega_0) = \frac{1}{d}\sum_{l=0}^{d-1}(2\overline{u_l}v_l), \quad L(\Omega_0) = \frac{1}{d}\sum_{l=0}^{d-1}\bar{u}_l^2,$$

and

$$N(\Omega_0) = \frac{1}{d}\sum_{l=0}^{d-1} v_l^2. \quad [37]$$

It is time also to remember that the eigenfunctions are determined only within an arbitrary phase and we take this into account. If $\exp(-j\psi_0/2)$ is the arbitrary phase factor applied to the $\vec{u}_{-j}$ eigenfunction, the above coefficients L and M are multiplied by $\exp(j\psi_0)$ and $\exp(-\psi_0)$ respectively, and we can write Eq. [36] in the form

$$\hat{X}(\Omega_0, p+i) = \cos(\chi)I + (-1)^i j\sin(\chi)O; \quad [38]$$

with

$$O(\Omega_0) = e^{j\psi_0}L + e^{-j\psi_0}N. \quad [39]$$

One cannot choose the coefficients I, L, N. They depend only on the frequency reference Ω_0, the sweep velocity Δ, and the rotation angle θ of the refocusing pulses used. According to Eq. [38] the signal generated by an initial transverse magnetization aligned with the reference axis x is constant along the echo train and given by I, while an initial transverse magnetization along the axis y will generate a signal changing sign every other echo with complex amplitude O. By doubling each phase-encoding value or, equivalently doubling the field of view along the phase-encoding direction, the two signals coming from the same geometric position will be separated. But to recover the two components of the object with equal accuracy and with an efficiency close to the one obtained in the CPMG case, one must have $|I| \simeq |O| \simeq \hat{X}_{\text{cpmg}}$, where $\hat{X}_{\text{cpmg}} = \sin(\theta/2)$.

3.4.5. Choosing ψ_0. O depends on ψ_0 which must be set. It is found that, for the large value of d that we will use, the values of $|I|$, $|L|$, $|N|$ do not vary substantially for different combs (different Ω_0). This is demonstrated by simulation in Fig. 4, where several comb values are superimposed. One possible explanation is that, once the frequencies Ω_l are mapped into one pixel with increasing resonant frequency, and if $d\theta^2 \gg 1$, the functions $u(\Omega_k)$, and $v(\Omega_k)$, form two very smooth functions in regard to the comb step $2\pi/d$. Then, on another comb, with Ω_0 distant from the first by less than the step $2\pi/d$, the eigenfunction is probably almost an interpolation of the first one and very close to it. That is not a demonstration, but this is the only justification we have for now. Assuming the relative independence of the eigenfunctions from Ω_0, one can demonstrate (and this is verified in Fig. 4) that $|L| = |N|$, by using a symmetry that the eigenfunction $\vec{u}_{-j}$ verifies at frequencies Ω and $\Omega + \pi$ (see Appendix C). If $|L| = |N|$, varying the phase ψ_0 will make the coefficient O describe a line segment in the complex plane; for only two values separated by π will the magnitude of this O coefficient be maximum.

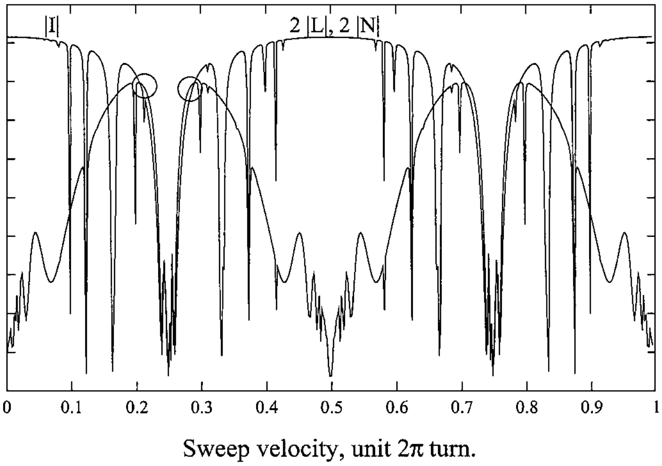

FIG. 4. The magnitude of the signal coefficients $|I|$, $|L|$, and $|N|$ for a sweep velocity $\Delta = 2\pi n/499$. $d = 499$ being a prime number, n has been varied from 1 to 499 without having to worry about common divisors between n and d. The rotation angle of the refocusing pulses was $\theta = 2$ radians (114.6°). Four different combs (Ω_0) have been simulated for each value of Δ and each coefficient. No significant difference is seen and the curves for different Ω_0 are superposed on each other. Also, the curves relative to L and N are indistinguishable from each other. Thus after suitable phasing of the eigenfunction the oscillating signal $|I|$ coefficient can be made equal to $|2L| = |2N|$. There are only two ranges of Δ values (four if one counts the negative counterparts) around $0.19 \times 2\pi$ and $0.31 \times 2\pi$ for which the constant signal and the oscillating signal are of same magnitude. Note that $\Delta = 0.25$ corresponds to the {XY − 4} (5) modulation and is probably the worst possible choice, at least in terms of signal magnitude, at this already small refocusing angle.

In practice this signal maximum is also attained when the choice of phase ψ_0, for all different combs, renders the function $u(\Omega)$, $v(\Omega)$ the smoothest. In turn, that is achieved when each comb phase $\psi_0/2$ is turned such that the resulting eigenvector $\exp(-j\psi_0/2) \times \vec{u}_{-j}(\Omega_l), l = 0, d-1$ is closest (in the sense of least squares) to the $-j$ eigenfunction of the perfect π case, $[1, 1]^t/\sqrt{(2)}$

$$e^{j\psi_0/2} = E_{uv}/|E_{uv}|, \quad E_{uv} = \sum_{l=0}^{d-1}(u_l + v_l). \quad [40]$$

As an example, Fig. 5 gives the eigenfunction $-j$ for the sweep velocity $\Delta = 2\pi \times 957/4999$ and for $\theta = 2$ radians, after having phased by Eq. [40] four different combs.

3.4.6. Choosing the sweep velocity Δ. Although the mathematical derivation is a bit involved, one must realize that the numerical computation of the $-j$ eigenfunction as written above, and the associated signals I, O, for a given θ and a given Δ, is very fast on modern computers, even for a denominator d very large. Very large d are necessary for small refocusing angles θ. The lower the refocusing angle, the more difficult it is to have quasi-equality between $|I|$ and $|O|$. Taking $d = 4999$ which is already a large number but also a prime, one can vary n and

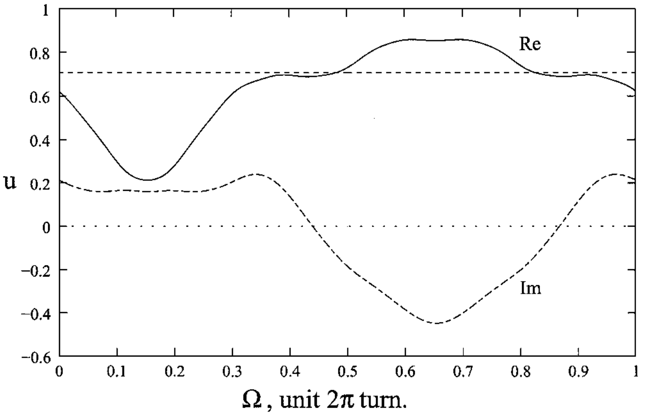

FIG. 5. One component (u) of the $-j$ eigenfunction for $\Delta = 2\pi \times 957/4999$ and $\theta = 2$ radians, at an even echo. Four different combs were used, each one with a phase factor adapted to minimize the mean square error with the $-j$ eigenfunction of the ideal $\theta = \pi$ case also shown on the drawing. The second component is deducible from the first by the $\Omega, \Omega + \pi$ symmetry exposed in Appendix C.

try to find its value(s) for which $|I|$ and $|O|$ are the closest to each other. One such value is $n = 957$, $\hat{\Delta} = 2\pi \times 957/4999$, or $2\pi \times 0.19144$, for which the values of $|I|$ and $|O|$ for a large range of refocusing angles θ are shown in Fig. 6. Other values are $n = 1543$ (corresponding approximately to $\pi - \hat{\Delta}$, or $\Delta = 2\pi \times 0.30866$) or $n = 3456$ (corresponding to $\pi + \hat{\Delta}$, or $\Delta = 2\pi \times 0.69134$). Obviously, the results for $n = 956$ or $n = 955$ are not very different when θ is greater than one radian, but differences appear at small refocusing angles. Admittedly, this is presently a rather futile discussion, as one needs first to be able to determine the preparation P in order to put the system into the desired subspace [see Eq. [35]], and as will be seen now, we succeeded in that only for θ above 2 radians. Still, it is useful to have a target eigenfunction for the preparation period giving suitable signals, even for small refocusing angles.

3.5. Preparation Period

In this subsection we will have to represent the rotation starting from thermal equilibrium, at each frequency Ω, $W_i(\Omega)$ in SU_2 form, at echo number i. Still, for notational simplicity, rather than following the evolution of the $W_i(\Omega)$ matrix, we will follow only its first column vector (or spinor, or CK parameters), noted $\vec{w}_i(\Omega)$. For d sufficiently large, one can consider now that we have defined $\vec{u}_{-j}(\Omega, \theta) = [u, v]^t$ continously for $0 < \Omega \leq 2\pi$, as shown in Fig. 5, and that for a sufficiently dense collection of θ (for instance $\theta = 3, 2.8, 2.6\ldots$ radians). Our aim is now to find a sequence of sweep velocities Δ_i, $i = 1\ldots p$ such that the recursion in Eq. [10], written in spinor form

$$\vec{w}_i(\Omega, \theta) = Q(c \exp(j\Omega), -js)S(\Delta_i)\vec{w}_{i-1}(\Omega, \theta),$$

(with $c = \cos(\theta/2)$, $s = \sin(\theta/2)$) gives a sequence of $\vec{w}_i(\Omega, \theta)$

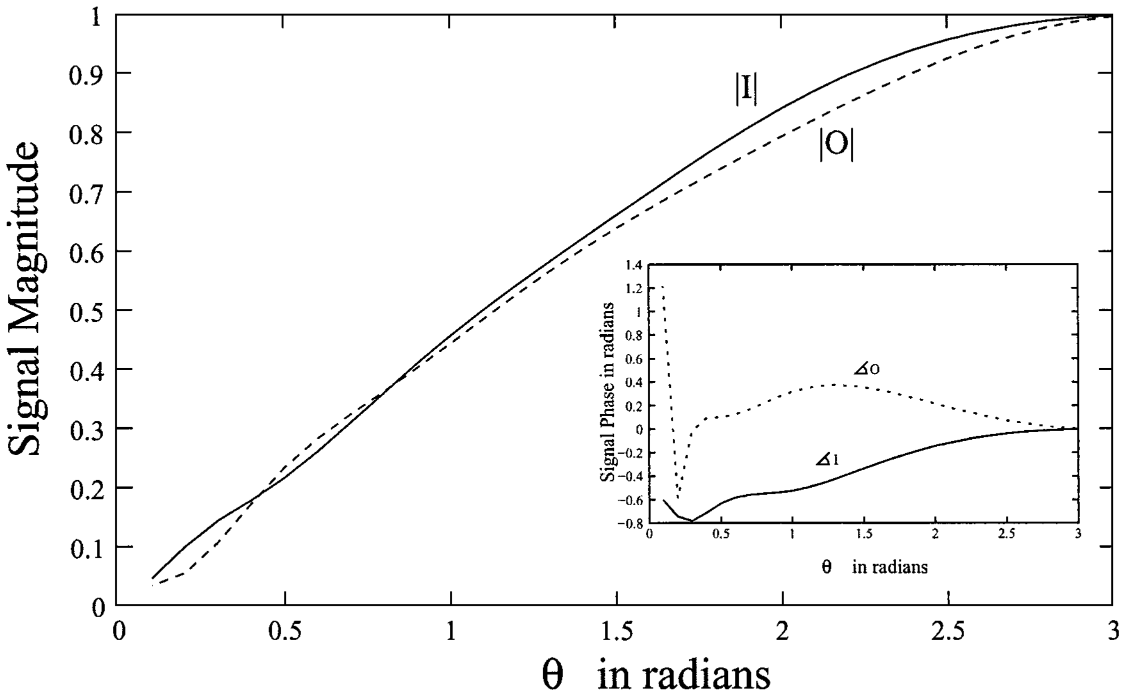

FIG. 6. The magnitude of the signal coefficients I and O for $n = 957$ and $d = 4999$ for $\theta = 0.1\ldots 3$ radians. There is a $\pm 3\%$ possible variation between $|I|$ and $|O|$. The inset represents the phase of the signal coefficients as a function of the nutation angle. We note that the phase is not null, but for nutation angles above 1.5 radian the phase variation is small (below 20°). Also, the two phase angles are almost opposite, i.e., the average of both phases is almost independent of θ.

which tends rapidly toward the eigenfunction $(-j)^i \vec{u}_{-j}(\Omega, \theta)$, when the initial condition is $w_0 = [1, 1]^t/\sqrt{2}$ (which is the spinor representation of the excitation pulse $R_y(\pi/2)$). The factor $(-j)^i$ is to take into account the natural evolution of the eigenfunction. Actually it is better to include that factor in the evolution equation of the state vector $\vec{w}_i$ itself, following the evolution of $j^i \vec{w}_i$ which ideally should tend toward the constant function $\vec{u}_{-j}$. Then Eq. [10] is replaced by

$$\vec{w}_i(\Omega, \theta) = jQ(c\exp(j\Omega), -js)S(\Delta_i)\vec{w}_{i-1}(\Omega, \theta), \quad [41]$$

and we can take

$$J(\Delta_i) = \sum_{i=1}^{p} g_i \sum_{\theta} \int_0^{2\pi} |\vec{u}_{-j} - \vec{w}_i|^2(\Omega, \theta)\, d\Omega$$

as the cost function to be minimized. g_i is a weight to more strongly emphasize the end of the preparation interval than its beginning. The internal quadratic term of the cost function is simplified, due to the normalization $|u_{-j}|^2 = |w_i|^2 = 1$, leaving only a term $\mathrm{Re}(\vec{u}^*_{-j}\vec{w}_i)$. But as both $j^i \vec{u}_{-j}$ and $\vec{w}_i$ are realizable by a $\pi/2$ excitation pulse followed by a train of echoes, they must satisfy a symmetry between Ω and $\Omega + \pi$ (see Appendix C), and one can show that $\int_0^{2\pi} \vec{u}^*_{-j} w_i \; d\Omega$ is real. Hence the cost function is simply

$$J(\Delta_i) = \sum_{i=1}^{p} g_i \sum_{\theta} \int_0^{2\pi} \vec{u}^*_{-j}\vec{w}_i(\Omega, \theta)\, d\Omega. \quad [42]$$

This is a problem of optimal control of a nonlinear system (although the cost function is linear). Still the system can be linearized and the optimization of the loss function performed by a classical quasi Newton procedure (*16*), without of course the guarantee that the global minimum is found.

3.6. Resulting Optimal Phase Modulation

The simplest result was obtained by optimizing the cost function of Eq. [42] for only $\theta = 2.8$ radians (and for $\hat{\Delta} = 2\pi \times 957/4999$). The preparation period was fixed to seven echoes, and with only a terminal loss function obtained by setting $g_i = 0$, $i = 1 \ldots 6$, $g_7 = 1$. The optimal parameters Δ_i are given in Table 1. The frequency sweep δ_i can be obtained by Eqs. [11] and [12]. Setting by convention the receiver phase at echo zero $\phi_{r0} = 0$, the first emission phase ϕ_{x1} is computed using Eq. [3]; and from that the receiver phase at the first echo ϕ_{r1} by Eq. [4]; and recursively all other emission and reception phases.

The in phase and out of phase component responses, calculated whether by the density matrix, or simply by the Bloch equations are shown in Fig. 7. Although the stabilization does not seem perfect, one can compare this result with what is obtained with the XY − 4 modulation (*5*) shown in Fig. 8. We see already that at 160° refocusing angle, the XY − 4 modulation does not give a sustained signal. Further, the spurious modulation is also very large. At 140° and below, the signals are chaotic. One advantage of the XY − 4 modulation may be that the out of phase signal and the in phase signal are identical, but clearly this modulation scheme should not be used below 170°.

TABLE 1
Optimal Sweep Velocities for a Preparation Period of Seven Echoes

i	Δ_i
1	0.191438
2	0.192650
3	0.225601
4	0.197626
5	0.129640
6	0.197671
7	0.282091
8...	957/4999

Note. Starting from the eighth interval, the sweep velocity becomes constant.

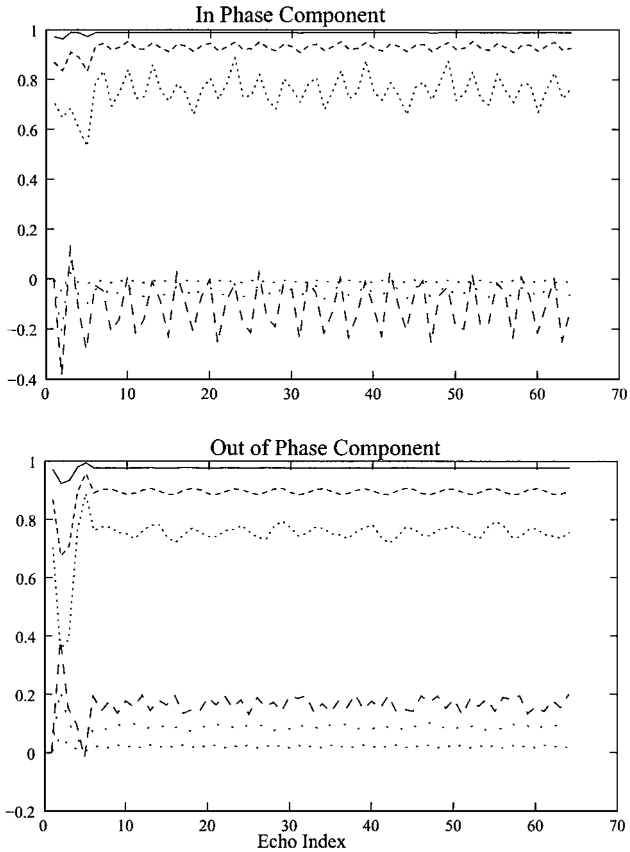

FIG. 7. The signals generated by an initial magnetization aligned, at echo zero, with x (in phase component) or by an initial magnetization aligned with y (out of phase component) for refocusing angles $\theta = 2.8, 2.4, 2.0$ radians (approximately 160, 140, 115°), with the sweep velocities given in Table 1. Both the real and imaginary parts of the signals are depicted.

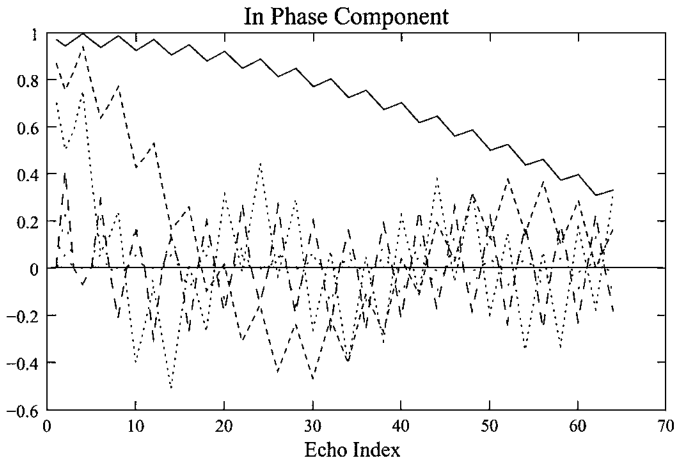

FIG. 8. The signals generated by the in phase component with the $XY-4$ modulation for refocusing angles $\theta = 2.8, 2.4, 2.0$ radians. The signals generated by the out of phase component are strictly identical.

We have tried other cost functions over a larger range of θ and longer preparation period. The results may be better than those presented here but only marginally so. Hence at present, the phase modulation given in Table 1 is our best choice. Also note that the modulation used in (*9*), and whose values are given in (*10*), is less effective than the modulation given here. Indeed that modulation was determined along with a possible refocusing angle modulation (i.e., amplitude and phase modulation of the RF train) of the first three RF pulses. When allowing an amplitude modulation, three pulses are even more efficient than the seven phase modulated pulses presented here and the signals are better stabilized. But in practice, making the refocusing angle vary, implies designing different refocusing pulses for each echo as we did in (*17*). This is not a simple task and in (*9*) we used only the phase modulation part making the result suboptimal. In future we may have to revert to amplitude and phase modulation but this needs careful design of RF pulses and also an assessment of the sensitivity with regard to B_1 inhomogeneity or calibration errors.

4. PRINCIPLE OF THE RECONSTRUCTION

The experimental result shown below is an imaging experiment and we have to explain, even if it is not in the most extreme details, how the echoes are used to form an image.

A sustained spin-echo experiment, as the one we have in hand, where one magnetization component stays constant and the other magnetization gives way to an alternating signal, imposes a reconstruction different from the classical 2DFT or 3DFT which is used with a CPMG experiment (*18*). In (*8*), we proposed to double each phase encoding: two consecutive echoes are acquired with the same encoding value. The reason why we must repeat the same phase encoding twice stems from another parameter we have not yet taken into account the "spurious receiver phase," which comes in addition to the object phase χ.

Let the initial transverse magnetization at echo zero, and in a given voxel, at any position in space $\xi = (x, y, z)$ and of volume $d\xi$, be $dX_0(\xi)$:

$$\begin{aligned} dX_0(\xi) &= \rho(\xi) e^{j\chi(\xi)}\, d\xi \\ &= (\rho_x(\xi) + j\rho_y(\xi))\, d\xi. \end{aligned}$$

Then the average magnetization in this voxel at echo $p+i$, and thus also the signal originating from the considered voxel would be, according to [38], and supposing that the signal coefficients $I(i)$ and $O(i)$ are stationary ($I(i) = I$, $O(i) = O$)):

$$d\hat{X}(p+i, \xi) = (I\rho_x(\xi) + (-1)^i \times Oj\rho_y(\xi))\, d\xi. \qquad [43]$$

But, actually we cannot identify this element of transverse magnetization with the signal that we receive from that element of volume. Indeed we must write this element of signal dS in the following manner:

$$\begin{aligned} dS(p+i, \xi) &= e^{j\varphi(\xi)}\, d\hat{X}(p+i, \xi) \\ &= e^{j\varphi(\xi)}(I\rho_x(\xi) + (-1)^i \times Oj\rho_y(\xi))\, d\xi. \qquad [44] \end{aligned}$$

The phase factor $\exp(j\varphi)$ is due to a possible phase error between the emission and the reception. Indeed we have defined the rotating frame uniquely in terms of the emitting field, and set the receiving phase in accordance to that field. But a spurious phase in the receiver may add itself to the signal phase. This could easily be corrected if this phase were a constant. But this phase can be dependent on the resonant frequency (due for instance to the antialiasing filter), and the correction may become more complicated. But more dramatically, the phase error φ can be space dependent, and particularly depends also on the position along the phase encoding direction(s). This is indeed the case if one uses two separated coils for emission and reception: $\varphi(\xi)$ then represents the angle between the B_1 receiving field (projected on the plane perpendicular to the main field) and the B_1 emitting field (also projected). Another source of possible space dependent phase error is the action of eddy currents during the read period of each echo. A priori, $\varphi(\xi)$ is varying slowly in space but it can easily account for some tens of degree variation in one field of view. We show now that this suffices to oblige us to repeat twice the same encoding value. For simplicity we consider a 2D image acquisition (not a 3D one). We suppose that we use a classical 2DFT encoding-reconstruction scheme with the encoding value increasing linearly between each echo: the increment between each encoding step is linked to the maximum extension of the field of view along the phase-encoding direction, that we denote by $2y_M$, and the magnetization at the positions $y = \pm y_M$ change sign every echo under the influence of that encoding. If now we consider Eq. [43] as representing the right model, we can deduce without writing any further equation that we will obtain the following reconstructed image ρ_r

$$\rho_r(\xi) = I\hat{\rho}_x(x, y) + Oj\hat{\rho}_y(x, y + \mathrm{y}_M), \qquad [45]$$

where $\hat{\rho}_x$ and $\hat{\rho}_y$ are low pass filtered versions of the two component of the object. There is aliasing between the position x, y and the position x, $y + y_M$. But one may argue that the two aliased values at the position $\xi = x, y$ are in quadrature to each other: $I\hat{\rho}_x(x, y)$ and $jO\hat{\rho}_y(x, y+y_M)$ and could be separated (if I and O are without phase, which is approximately the case according to Fig. 6). Unhappily the phase factor $\exp(j\varphi(\xi))$ renders this idea inapplicable. It is easily found that, with the true signal expression [44] and still with a classical 2DFT phase encoding scheme, the reconstructed image becomes

$$\rho_r(\xi) = I\hat{\rho}_x(x, y)e^{j\varphi(x,y)} + Oj\hat{\rho}_y(x, y + \mathrm{y}_M)e^{j\varphi(x,y+y_M)}. \qquad [46]$$

It suffices here to consider the degenerate case: it corresponds to the case where the spurious receiver phase at position x, y and the spurious receiver phase at position x, $y + y_M$, have a difference of $\pi/2$! Then the two magnetization elements $\hat{\rho}_x(x, y)$ and $\hat{\rho}_y(x, y + y_M)$ emit two indistinguishable signals (if the signal coefficients I and O have the same phase; otherwise one could find another value for the receiver phase difference) and thus the two magnetizations are irremediably aliased.

To counteract that, we use the same phase encoding for two subsequent echoes i and $i + 1$, then adding and substracting the two signals, $S_1(p + i) = S(p + i + 1) + S(p + i)$, $S_2(p + i) = S(p + i + 1) - S(p + i)$, we obtain from [44]

$$\begin{aligned} dS_1(p + i) &= 2Ie^{j\varphi(\xi)}\rho_x(\xi)\,d\xi, \\ dS_2(p + i) &= 2jOe^{j\varphi(\xi)}\rho_y(\xi)\,d\xi. \end{aligned} \qquad [47]$$

Because the two quantities $2I\exp(j\varphi(\xi))\rho_x(\xi)$ and $2O\exp(\varphi(\xi))\rho_y(\xi)$ have a phase which is varying in space very slowly we can acquire and reconstruct each one, separately, by a half k-space (homo-dyne) acquisition-reconstruction (*19*).

5. EXPERIMENTAL RESULT AND DISCUSSION

The principle of the quadratic phase modulation, with a well chosen sweep velocity Δ and after a suitable preparation, has now been validated. It has been used on volunteers not only for spine diffusion imaging (*9*), but also for diffusion tensor imaging in the head (*20*), T_2 and T_2^* sensitization (*21*), and even for phase shift thermometry (*22*). Admittedly these studies were not used with the new preparation period presented here. The effectiveness of this new modulation scheme is verified here on a phantom and this is depicted in Fig. 8. This is a single shot acquisition, where we have forcibly induced a phase variation in the object by inserting a gradient blip between the excitation RF pulse and the echo 0 reference time. Another side of the development which is worth mentioning has been the evolution of the reconstruction program. In the first volunteer study (*9*) we performed the half k-space acquisition the usual way, and the number of k-lines acquired in the complementary half k-space was relatively high (6 to 8 over-scan lines) in order to identify the spurious receiver phase φ. As we must double each phase encoding, this resulted in a much delayed minimum echo time (12 to 16 echo spaces, or in the order of 60 to 90 ms increase in minimum echo time on a whole body scanner), and the gain in signal to noise ratio compared to the Alsop solution (*4*) was often entirely lost, due to T2 relaxation. The other studies, including the present one, have used an acquisition–reconstruction scheme, where an independent acquisition permits characterizing the receiver phase φ with very good precision (actually using a full k-space acquisition); then during the normal acquisition, we just have to correct for the phase φ, not characterize it, and the number of over-scan lines can be reduced to 2. Still, if this allows us to retain a substantial portion of the signal coming from short T2 species, it does not reduce the blurring associated with the signal slope in the k-space (see Fig. 9 caption). Against this, the only two options are using multishot acquisition or finding a way to traverse the k-space more rapidly. But, at least for diffusion imaging, multishot acquisition is only feasible if this acquisition can acquire and compensate for the random, shot dependent, object phase modulation χ. Recently Pipe *et al.* (*23*) proposed such a self-navigated acquisition. Let us note that these authors used an *XY* quadratic phase. They did not double each phase encoding and their scheme will probably not work, as is, with distinct emitting–receiving coils. Another route that may be envisioned is to couple the here presented non-CPMG acquisition–reconstruction scheme with coil sensitivity encoding (*24*). Normally, once the two real and imaginary responses

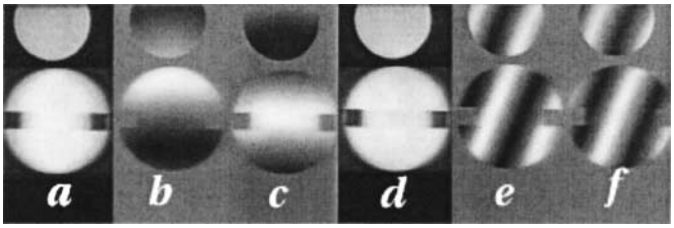

FIG. 9. Magnitude (a, d), real part (b, e), and imaginary part (c, f) of an object with a variable phase χ that we have induced by a gradient blip interposed between the excitation pulse and echo zero. This gradient was applied along the phase encoding direction, left to right. In (a, b, c) this gradient was nulled and in (d, e, f) its amplitude was calculated to induce, approximately, a $2 \times 2\pi$ phase variation in the field of view (FOV) extension along y. The phase variation in (a, b, c) may come from a spurious object phase χ, but may as well, and more probably, be due to a receiver phase φ. This is the reason why we show two acquisitions, one acquisition not being sufficient proof of the efficiency of the RF phase modulation to obtain independency with respect to the initial phase of the magnetization. The acquisition was single-shot. The FOV was $260 \times 130\{\mathrm{mm}\}^2$ for a matrix 256×64. Two over-scan lines were acquired and thus a total of $(32 + 2) \times 2$, or 68 echoes were acquired, lasting around 340 ms. The object at bottom has a T_2 on the order of 200 ms, whereas the top cylinder is filled with tap water with a T_2 greater than 1500 ms. Note the higher blurring of the reconstructed short T_2 object in comparison to the water cylinder.

are separated by the double acquisition as in Eq. [47], the sensitivity encoding scheme can be applied independently.

6. CONCLUSION

We presented the mathematical basis for the use of quadratic phase modulation to generate a train of spin echoes which is almost completely insensitive to initial phase. Although not yet perfect this approach has already proven useful in practice. Future progress may rely on the background presented here.

APPENDIX A

Let $\mathcal{R}(\Omega_d) = Q(\alpha, \beta)$. We separate the real part α_r of $\alpha = \alpha_r + j\alpha_i$, and put $-j$ as a factor of a vector matrix $\mathcal{R}(\Omega_d) = \alpha_r \mathbf{1} - j\sigma(-\alpha_i, j\beta)$. Then we normalize the vector part, in order to put the rotation matrix in the form of Eq. [26]. For that, we define $\mathcal{S} = \sqrt{|\beta|^2 + \alpha_i^2}$ and write $\mathcal{R}(\Omega_d) = \alpha_r \mathbf{1} - j\mathcal{S}\sigma(z, X)$ with $z = -\alpha_i/\mathcal{S}, X = j\beta/\mathcal{S}$. Then we factor the vector matrix $\sigma(z, X)$ in $U\sigma_z U^*$, with $U = Q(u, v)$. We must verify simultaneously: $|u|^2 - |v|^2 = z, 2\bar{u}v = X$. This is a classical problem which leaves a phase indetermination (linked to the phase indetermination of an eigenvector); we choose to have the first component real and find $u = \sqrt{(1+z)/2}$, $v = X/\sqrt{2(1+z)}$.

APPENDIX B

The Z transform approach, in addition to the SU_2 formalism, is the basis of the so-called Shinnar–LeRoux algorithm (*15*). We want to find the cycle rotation matrix, but this time for any frequency Ω, not just for one comb of frequencies. We show that we can express this rotation in terms of polynomials of the variable $Z = \exp(j\Omega)$. We write Eq. [10] using the SU_2 representation of the central rotation by Eq. [27] but using Z as notation for $\exp(j\Omega)$

$$\mathcal{R}_i(\Omega) = Q(cZ, -js)\mathcal{R}_{i-1}(\Omega - \Delta),$$

for $i = 1 \ldots .$

With the initial condition, at echo zero $R_0 = Q(1, 0)$. We need only follow the first column α_i, β_i (or Cayley–Klein parameters) of the SU_2 rotation matrix $\mathcal{R}_i(\Omega) = Q(\alpha_i(\Omega), \beta_i(\Omega))$. Let us develop the two components $\alpha_i(\Omega), \beta_i(\Omega)$ in a series, possibly infinite, of powers of Z (or otherwise stated, take the inverse Fourier transform of these two quantities): $\alpha_i(\Omega) = \sum_{k=-\infty}^{\infty} \alpha_{i,k} Z^{-k}$, $\beta_i(\Omega) = \sum_{k=-\infty}^{\infty} \beta_{i,k} Z^{-k}$. The shift in frequency by Δ is simply expressed in terms of the coefficients $\alpha_{i,k}$, $\beta_{i,k}$ by a local multiplication by the power of $w = \exp(j\Delta)$, $\alpha'_{i,k} = w^k \alpha_{i-1,k}$, $\beta'_{i,k} = w^k \beta_{i-1,k}$. The central rotation part involves a shift by plus one or minus one and a combination: $\alpha_{i,k} = c\alpha'_{i,k+1} - js\beta'_{i,k}$, $\beta_{i,k} = -js\alpha'_{i,k} + c\beta'_{i,k-1}$. We then arrive at a recursion on the time domain coefficient very similar to the one used in the (direct) SLR transformation. This way, one can simulate very precisely the response of the system even for an arbitrary phase modulation (i.e., even when Δ is variable with i).

But for now, we use these equations for obtaining some very simple results. We have the initial condition $\alpha_{0,k} = 0 k \neq 0$, $\alpha_{0,0} = 1$, $\beta_{0,k} = 0\, \forall k$. Without entering the detail of the calculus let us make one or two steps of the just defined recursion, only to discover what coefficients are not zero. At echo 1 we have: $\alpha_1(Z) = cZ$, $\beta_1(Z) = \beta_{1,0}$. At echo 2 the polynomials have the form $\alpha_2(Z) = c^2 Z^2 + \alpha_{2,0}$ and $\beta_2(Z) = \beta_{2,1} Z^{-1} + \beta_{1,-1} Z$. One can infer, and verify by recurrence that at echo number i, $\alpha_i(Z) = Z^i a(Z^{-2})$, $\beta_i(Z) = Z^{i-1} b(Z^{-2})$, where a, b are polynomials of order $0 \ldots i-1$ of the variable Z^{-2}.

We have some interesting symmetry properties to derive from this. At even echo i number, α has only even order powers of Z, whereas β has only odd order powers, these properties being swapped at odd echo. The consequence in the frequency domain is that, for echos with even index, $\alpha_{2p}(\Omega + \pi) = \alpha_{2p}(\Omega)$, whereas $\beta_{2p}(\Omega) = -\beta_{2p}(\Omega + \pi)$. Conversely, at odd echo number $N = 2p + 1$, the symmetry is reversed $\alpha_{2p+1}(\Omega + \pi) = -\alpha_{2p+1}(\Omega)$, $\beta_{2p+1}(\Omega) = \beta_{2p+1}(\Omega + \pi)$.

Another easy result is the value of the "leading term" $\alpha_{N,-N}$ (also the zero order coefficient of a). In the recursion it is never combined with any other coefficient and its value at echo N is easily derived: $\alpha_{N,-N} = c^N \times (1 \times w \cdots \times w^{N-1})$, or $\alpha_{N,-N} = c^N \exp(j\Delta N(N-1)/2)$.

If we now consider one complete cycle of the modulation, $N = d$, if $\Delta = 2\pi n/d$, one has simply $\alpha_{d,-d} = c^d$, with, we recall, $c = \cos\theta/2$. Consider now the real part of $\alpha_d(Z)$, but seen in the Ω domain. Recall that according to the spinor representation this is equal to $\cos(\Theta(\Omega)/2)$, where $\Theta(\Omega)$ is the rotation angle of the cycle rotation. This cycle rotation is the same for all the frequency positions belonging to the same comb $\Omega = \Omega_0 + l \times 2\pi n/d$. Hence $\mathrm{Re}(\alpha_d(\Omega))$ is, in the angular frequency domain, a periodic function of period $2\pi/d$; otherwise stated, in the time domain, $\mathrm{Re}(\alpha_d(Z))$ contains only coefficients of powers Z^d: $\ldots Z^{2d}, Z^d, 1, Z^{-d}, Z^{-2d}, \ldots$. But $\alpha_d(Z)$ has terms of Z powers only in the range $d, -d+1$. Hence $\mathrm{Re}(\alpha_d(z))$ can only be in the form $(c^d Z^d + \alpha_0 + c^d Z^{-d})/2$ or $\mathrm{Re}(\alpha_d(Z)) = c^d \cos(d\Omega) + \alpha_0$.

Consider now d odd. In this case the comb relative to Ω_0 and the comb relative to $\Omega_0 + \pi$ are distinct and interlaced (separated by π/d), and also, according to the symmetry property, $\alpha_{2p+1}(\Omega + \pi) = -\alpha_{2p+1}(\Omega)$, so $\mathrm{Re}(\alpha)$ changed sign between the two combs, and thus the constant term α_0 must be null.

Finally, the cycle rotation angle is given by $\cos(\Theta(\Omega)/2) = c^d \cos(d\Omega)$. For d sufficiently large c^d is very small, even when c is large (θ small), and thus the cycle rotation angle Θ is close to π.

APPENDIX C

From Appendix B we know that the Cayley–Klein parameters (or spinor components) α_i, β_i of the integrated rotation $\mathcal{R}_i$ have symmetry relations between the position frequency Ω and the frequency $\Omega + \pi$. This has been derived with $\Delta =$ constant but it is true for any variable sweep velocity. Let us consider W_i the rotation from thermal equilibrium to echo i; we must include $R_y(\pi/2)$ in front of $\mathcal{R}_i$ and even a rotation $R_z(\chi)$, giving $W_i = \mathcal{R}_i R_z(\chi) R_y(\pi/2)$. Let $\alpha_{-,i}$, $\beta_{-,i}$ be the CK parameters of W_i. A simple calculation shows that the symmetry on α_i, β_i induces the following symmetry: $\alpha_{-,i}(\Omega+\pi) = \bar{\beta}_{-,i}(\Omega)$, $\beta_{-,i}(\Omega+\pi) = \bar{\alpha}_{-,i}(\Omega)$, if i is even. If the echo number is odd the same relations apply but with a minus sign added. The important point is that the eigenfunction $\vec{u}_{-j}(\Omega)$ must satisfy these symmetries if we want to be able to attain it by $\pi/2$ excitation followed by a train of echoes!

We now verify that it is indeed the case, at least when d is odd. When d is odd, if $\mathcal{R}_d(\Omega)$ is in term of its CK parameters, $\mathcal{R}_d(\Omega) = Q(\alpha_d, \beta_d)$, then $\mathcal{R}_d(\Omega+\pi) = Q(-\alpha_d, \beta_d)$. And one verifies that if $\mathcal{R}_d(\Omega)$ can be decomposed in eigenvectors by $Q(\alpha_d, \beta_d) = U\Lambda U^*$ with $U = Q(u, v)$ then $\mathcal{R}_d(\Omega+\pi)$ can be decomposed in $Q(-\alpha_d, \beta_d) = V(-\bar{\Lambda})V^*$ with $V = Q(\bar{v}, \bar{u})$. Also as the eigenvalues of the cycle rotation are, for d sufficiently large, close to $j, -j$, we can choose for the eigenfunction relative to $-j$, on the comb Ω_0: $\vec{u}_{-j}(\Omega_0) = [u, v]^t$ and on the comb $\Omega_0 + \pi : \vec{u}_{-j}(\Omega_0+\pi) = [\bar{v}, \bar{u}]^t$. Thus realizing a spinor function which corresponds to an even echo attainable from thermal equilibrium (we say realizable). If we apply a phase factor $\exp(-j\varphi_0/2)$ on one comb, we will apply $\exp(j\varphi_0/2)$ on the other comb to keep the consolidated function $\vec{u}_{-j}(\Omega)$ realizable. Note that the evolution of $\vec{u}_{-j}$ from echo to echo being reduced to a simple multiplication by $-j$, at the next echo (odd echo), the other type of symmetry is automatically obtained. Also, the symmetry on the CK parameters (spinor component) induces the following symmetry on the magnetizations coefficients: $I(\Omega_0+\pi) = I(\Omega_0)$, $L(\Omega_0+\pi) = L(\Omega_0)$, $N(\Omega_0+\pi) = N(\Omega_0)$.

ACKNOWLEDGMENTS

The author thanks Graeme McKinnon, first, for having made him aware some years ago of the shortcomings of the CPMG sequence, second, for having designed a programming environment that the author has used exclusively for all his developments and particularly the practical application of the present work, and finally for having corrected the present article.

REFERENCES

1. J. Hennig, Multiecho imaging sequences with low refocusing angles, *J. Magn. Reson.* **78,** 397–407 (1988).
2. H. Y. Carr and E. M. Purcell, Effects of diffusion on free precession in nuclear magnetic resonance experiments, *Phys. Rev.* **94,** 630–638 (1954).
3. D. G. Norris, P. Bornert, T. Reese, and D. Leibfritz, On the application of ultra-fast rare experiments, *Magn. Reson. Med.* **27,** 142–164 (1992).
4. D. C. Alsop, Phase insensitive preparation of single-shot rare: Application to diffusion imaging in humans, *Magn. Reson. Med.* **38,** 527–533 (1997).
5. A. A. Maudsley, Modified Carr–Purcell–Meiboom–Gill sequence for NMR Fourier imaging applications, *J. Magn. Reson.* **38,** 527–533 (1997).
6. T. Gullion, D. B. Baker, and M. S. Conradi, New, compensated Carr–Purcell sequences, *J. Magn. Reson.* **89,** 479–484 (1990).
7. J. B. Murdoch, An "Effective" method for generating spin-echo intensity expression, *in* "Proceedings Second SMR Scientific Meeting," San Francisco, CA, 1994, p. 1145.
8. P. Le Roux, Progress in non-CPMG fast spin echo, *in* "Proceedings Seventh ISMRM Scientific Meeting," Philadelphia, PA, 1999, p. 7.
9. P. Le Roux, C. A. Clark, A. Darquie, and P. G. Carlier, Rapid diffusion imaging of the human spinal cord using non-CPMG single shot fse, *in* "Proceedings Eighth ISMRM Scientific Meeting," Denver, CO, 2000, p. 1501.
10. P. Le Roux, Spin echoes with a quadratic phase modulation of the rf pulse train, *in* "Proceedings Ninth ISMRM Scientific Meeting," Glasgow, UK, 2001, p. 1788.
11. A. Messiah, "Quantum Mechanics," Dover, New York (2000).
12. E. Jaynes, Matrix treatment of nuclear induction, *Phys. Rev.* **98,** 1099–1105 (1954).
13. A. L. Bloom, Nuclear induction in inhomogeneous fields, *Phys. Rev.* **98,** 1105–1111 (1954).
14. B. Blumich and H. W. Spiess, Quaternions as a practical tool for the evaluation of composite rotations, *J. Magn. Reson.* **61,** 356–362 (1985).
15. J. Pauly, P. Le Roux, D. Nishimura, and A. Macovski, Parameter relations for the Shinnar–Le Roux selective excitation pulse design algorithm, *IEEE Trans. Med. Imaging* **10,** 53–65 (1991).
16. J. Nocedal and S. Wright, "Numerical Optimization," Springer-Verlag, New York (1999).
17. P. Le Roux and S. Hinks, Echoes stabilization in fse sequences, *Magn. Reson. Med.* **30,** 183–191 (1993).
18. P. Le Roux and G. McKinnon, Non cpmg fast spin echo with full signal, *in* "Proceedings Sixth ISMRM Scientific Meeting," Sydney, Australia, 1998, p. 574.
19. D. Noll, D. Nishimura, and A. Macovski, Homodyne detection in magnetic resonance imaging, *IEEE Trans. Med. Imaging* **10,** 154–163 (1991).
20. M. E. Bastin and P. Le Roux, Application of non-cpmg fast-spin-echo sequences to mr diffusion imaging, *in* "Proceedings Ninth ISMRM Scientific Meeting," Glasgow, UK, 2001, p. 1548.
21. P. Carlier, A. Darquie, V. Lebon, A. Andrade, R. J. Gilles, and P. Le Roux, Detection of tissue oxygen changes using interleaved t2 and t2* non-cpmg ssfse imaging, *in* "Proceedings Ninth ISMRM Scientific Meeting," Glasgow, UK, 2001, p. 905.
22. M. V. anf Frank Lethimonnier, P. Pattynama, and P. Le Roux, Use of fast-spin-echo mri for mr thermometry, *in* "Proceedings Ninth ISMRM Scientific Meeting," Glasgow, UK, 2001, p. 2195.
23. J. G. Pipe, V. G. Farthing, and K. P. Forbes, Multishot diffusion-weighted fse using propeller mri, *Magn. Reson. Med.* **47,** 42–52 (2002).
24. K. Pruessman, M. Weiger, M. Scheidegger, and P. Boesiger, Sense: Sensitivity encoding for fast mri, *Magn. Reson. Med.* **42,** 952–962 (1999).